SLIPPAGE IN DYNAMIC SYSTEMS WITH COLLISION INTERACTIONS

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The distinctive features of the phase space of a multidimensional dynamic system with collision interactions are investigated under the usual idealizations. The existence of certain configurations of the phase space is established. Motions near these configurations (called "slippage" motions) constitute convergent infinite sequences of collision-collisionless domains. The relationship between the states of the system at the beginning and end of slippage can be obtained by introducing a new dynamic model of systems with dynamic interactions. This model permits absolutely inelastic collisions to occur in the above configurations followed by the motion of the masses involved in the collisions under a kinematic constraint; collisions occurring outside these configurations cannot be completely inelastic and have a velocity restitution coefficient different from zero.

This model of slippage is a generalization of the idealization of collision interactions proposed in [1] on the basis of an experimental study of interactions in a chronometer. This idealization takes account of two collisions: the not completely elastic first collision and the second inelastic collision with subsequent motion under a kinematic constraint.

This "improved" dynamic model of systems with collision interactions made possible the investigation of complex periodic and nonperiodic modes of operation through analog and digital computer simulation [2 and 3].

1. Let the collisionless motions of the two masses of a system between which collision interactions can occur be described in dimensionless form by Eqs.

$$x_1^{\cdot\cdot} = F_1(x_1, x_1^{\cdot}, \ldots, x_n^{\cdot}, t) + F_{12}(y, y^{\cdot}), \qquad \mu x_2^{\cdot\cdot} = F_2(x_1, x_1^{\cdot}, \ldots, x_n^{\cdot}, t) - F_{12}(y, y^{\cdot})$$

$$y = x_2 - x_1 > 0 \tag{1.1}$$

Here x_i are the phase coordinates of the system and x_1 and x_2 denote the displacements of the colliding masses; F_{12} is the interaction force between the masses in the time intervals between collisions: F_1 and F_2 include all of the remaining forces acting on the above masses; μ is the ratio of the masses.

The relative mass displacement $y = x_2 - x_1$ is given in accordance with (1.1) by Eq.

$$\mu y^{\cdot\cdot} = F_2 - \mu F_1 - (1 + \mu)F_{12} \tag{1.2}$$

The collision occurs on the surface Π, $y = 0$, and is idealized as an instantaneous change in the velocities $y^{\cdot}$, $x_1^{\cdot}$ in accordance with the known relations

$$y^{\cdot +} = -Ry^{\cdot -}, \qquad x_1^{\cdot +} = x_1^{\cdot -} + \frac{\mu(1+R)}{1+\mu} y^{\cdot -} \tag{1.3}$$

where $y^{\cdot -}$, $x_1^{\cdot -}$ and $y^{\cdot +}$, $x_1^{\cdot +}$ represent the precollision and postcollision velocities and R is the velocity restitution coefficient for the collision $(0 \leqslant R < 1)$.

Let us investigate the behavior of the phase trajectories in the neighborhood of the collision interaction surface Π. This surface can be broken down into two parts: Π_1 ($y = 0$, $y^{\cdot} > 0$) on which the trajectories of the collisionless motions which enter the half-space G ($y > 0$) originate, and Π_3 ($y = 0$, $y^{\cdot} < 0$) on which the collision interaction trajectories begin. The behavior of the trajectories in the neighborhood of the boundary of the half-surfaces Π_1 and Π_3 ($y = 0$, $y^{\cdot} = 0$) is determined by the sign of the second derivative $y^{\cdot\cdot}$ of the

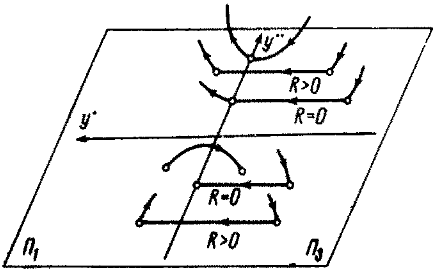

Fig. 1

relative mass displacement and by the quantity R which determines whether the collision is inelastic or partially elastic. Fig. 1 shows the four possible trajectories,

$$y'' > 0, \quad R = 0, \qquad y'' > 0, \quad R > 0,$$
$$y'' < 0, \quad R = 0, \qquad y'' < 0, \quad R > 0$$

In the first two cases the phase point leaves the boundary $y = 0$, $y' = 0$ and enters the half-space G.

In the third case ($y'' < 0$, $R = 0$) the collision results in adhesion of the masses, after which they move under a kinematic constraint for a finite time interval. This fact requires us to supplement the definition of dynamic system (1.1) by the addition of a certain force Q which represents the interaction between the kinematically constrained surfaces. During the above motion Q varies in such a way that the condition $x_1 \equiv x_2$ is fulfilled. Hence, the Eqs. of motion of the masses can be written as

$$x_1'' = F_1 + F_{12} + Q, \qquad \mu x_2'' = F_2 - F_{12} - Q, \qquad x_1 \equiv x_2, \qquad Q < 0$$

or, after eliminating Q, as

$$(1 + \mu)\, x_1' = F_1 + F_2, \qquad y = 0, \qquad F_2 - \mu F_1 - (1 + \mu) F_{12} < 0 \tag{1.4}$$

In the fourth case ($y'' < 0$, $R > 0$) the phase point generally cannot fall on the boundary $y = 0$, $y' = 0$. However, it is precisely this case which is of the greatest interest.

2. Let us show that there exists some neighborhood of the boundary $y = 0$, $y' = 0$, $y'' < 0$ which has the following property: when the phase trajectory enters this neighborhood, subsequent motion is accompanied by an infinite sequence of collision interactions and brings the phase point ever closer to the state

$$y = 0, \qquad y' = 0, \qquad y'' = 0, \qquad y''' > 0$$

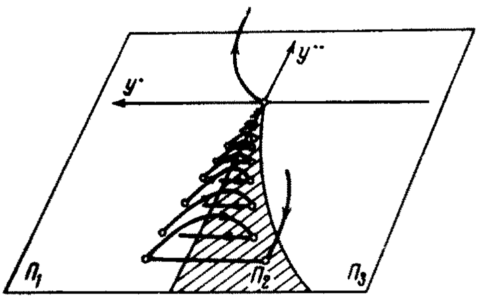

Fig. 2

from which it leaves the indicated neighborhood (Fig. 2) and enters the half-space G. We shall call such motion "slippage" and the corresponding portion Π_2 of the collision interaction surface the "slippage plate".

At the instant t_0 let the phase point M_0 enter some domain Π_2 of the surface $y = 0$ bounded by the conditions

$$0 < Y_* < y'''(t) < Y, \qquad \frac{9R - 5}{12R} < \frac{Y_*}{Y} \tag{2.1}$$

$$0 < -y''(t) < \frac{2Yt_*}{3 - \sqrt{4 + 5R}}, \qquad 0 < \frac{-y'(t)}{[y''(t)]^2} < \frac{5(1 - R)}{24RY} \tag{2.2}$$

We assume that the solution $y(t_0 + t)$ of Eq. (1.2) characterizing the collisionless motion has properties sufficient to permit its representation as a Taylor series for $t < t_*$.

The arrival of the trajectory at the point M_0 is accompanied by a collision (1.3) which is followed by the usual portion of the motion in the half-space G. Its duration is defined as the positive root τ_1 of Eq. $y(t_0 + t) = 0$ which for $t < t_*$ can be written as

$$t\,[-Ry'(t_0) + {}^1/_2 t^2 y''(t_0) + {}^1/_6 t^2 y'''(t_0 + \theta t)] = 0 \qquad (0 < \theta < 1) \tag{2.3}$$

Since the smallest positive root of (2.3) increases with an increasing derivative $y'''(t_0 + \theta t)$, and since restrictions (2.1) and the second restriction of (2.2) apply, it follows that

$$\tau_1 < -\frac{y_0''}{2Y}(3 - \sqrt{4 + 5R}) < t_* \qquad \left(\tau_1 \to 0 \text{ при } \frac{Ry_0'}{(y_0'')^2} \to 0\right) \tag{2.4}$$

The relative velocity and acceleration at the point M_1 immediately before the next collision are

$$y_1' = y'(t_0 + \tau_1) = -Ry_0' + \tau_1 y_0'' + {}^1/_2 \tau_1^2 y'''(t_0 + \theta_1 \tau_1) \tag{2.5}$$

$$y_1'' = y''(t_0 + \tau_1) = y_0'' + \tau_1 y'''(t_0 + \theta_2 \tau_1) \qquad (0 < \theta_1,\ \theta_2 < 1) \tag{2.6}$$

From (2.6) it follows that $\ddot y_1 > \ddot y_0$, so that the relative acceleration has approached zero. Let us show, however, that the point M_1 also belongs to the slippage plate, i.e. that $\ddot y_1$ and $\dot y_1/(\ddot y_1)^2$ satisfy inequalities (2.2).

a) The validity of $-\ddot y_1 > 0$ follows from (2.6), (2.1), and (2.4),

$$-\ddot y_1 > -\ddot y_0 - \tau_1 Y > -\tfrac{1}{2}\ddot y_0 (\sqrt{4+5R} - 1) > 0 \tag{2.7}$$

b) The restriction imposed by inequality (2.2) on the largest value of $-\ddot y_1$ is fulfilled by virtue of (2.6), since $-\ddot y_1 < -\ddot y_0$.

c) The validity of $-\dot y_1 > 0$ follows from (2.5), (2.1), (2.7), and (2.3),

$$-\dot y_1 > (R\dot y_0 - \tfrac{1}{2}\tau_1 \ddot y_0) + \tfrac{1}{2}\tau_1(-\ddot y_0 - \tau_1 Y) > R\dot y_0 - \tfrac{1}{2}\tau_1 \ddot y_0 > 0$$

d) It remains for us now to verify fulfillment of the condition

$$-\frac{\dot y_1}{(\ddot y_1)^2} < \frac{5(1-R)}{24RY} \tag{2.8}$$

In accordance with (2.6), (2.1), and (2.3) we have (2.9)

$$-\frac{\dot y_1}{(\ddot y_1)^2} < \frac{-Rq + p - \frac{1}{2}p^2 Y_*}{(1-pY)^2} < \frac{3p + p^2(Y - 3Y_*)}{6(1-pY)^2} \qquad \left(q = -\frac{\dot y_0}{(\ddot y_0)^2},\ p = -\frac{\tau_1}{\ddot y_0}\right)$$

Let us replace condition (2.8) which we are in the process of verifying by a stricter condition in accordance with (2.9). We arrive at the inequality

$$\frac{5+R}{12R}\left[\frac{5(1-R)}{5+R} - 2pY\right] + p^2Y^2\left(\frac{Y_*}{Y} + \frac{5-9R}{12R}\right) > 0 \tag{2.10}$$

This inequality is fulfilled since the expression in parentheses in the second term is positive by virtue of (2.1), while the expression in brackets is positive by (2.4),

$$2pY < 3 - \sqrt{4+5R}$$

Thus, on entering the configuration Π_2, the phase point finds itself in the slippage state whence it emerges only on reaching the "edge" of the slippage plate where $y = \dot y = \ddot y = 0$.

As we see from the above analysis, the conditions (2.1) and (2.2) adopted in proving the existence of the slippage plate do not define exactly the boundaries of this plate. It can be readily shown, however, that such a boundary does, in fact, exist. This is because the small neighborhood $y = \dot y = \ddot y = 0$, $\dddot y > 0$ contains not only the points where slippage begins, but also the points where trajectories leaving this domain originate. This is precisely the property shared by the set of points $\dot y < 0$, $\ddot y = 0$. Eq. (2.3) has no positive roots in this case regardless of how small $|\dot y|$ might be.

3. As the time intervals T_i between collisions diminish, the slippage phase trajectory approaches the trajectory of the masses moving under the kinematic constraint $y = \dot y = 0$. The required condition for slippage $\ddot y < 0$, or, by (1.2),

$$F_2 - \mu F_1 - (1 + \mu)F_{12} < 0 \tag{3.1}$$

will differ less and less from the similarly written condition of mass motion under kinematic constraint (1.4). We note that if the forces F_1, F_2 and F_{12} are functions of time alone, then conditions (3.1) computed for slippage and motion under the kinematic constraint coincide completely.

These considerations enable us to recommend the following idealized dynamic model of systems with collision interactions for describing slippage: the slippage configurations in the phase spaces of the above systems admit of inelastic collisions followed by motion of the masses involved in the collision under a kinematic constraint; outside these configurations there occur only partially elastic collisions with a velocity restitution coefficient different from zero.

The size of Π_2 depends on R and diminishes as $R \to 1$. In accordance with (2.1) and (2.2) estimation of the size of Π_2 requires knowledge of the time interval of $\dddot y(t)$ variation and the convergence radius t_*. In the study of uncomplicated systems with a three-dimensional phase space this presents no difficulties, since a single trajectory emerges from the edge of the plate and since t_* is sufficiently large. The study of complex systems, on the other hand, requires a computer which enables one to "catch" the point of origin of the slippage on the basis of sufficient closeness of its trajectory to the trajectory of motion under a kinematic constraint.

The above analysis was carried out for systems with a single collision pair. The phase spaces of systems with several collision pairs contain a slippage plate for each pair.

4. As an example, let us determine the size of the slippage plate in the phase space y, $y^{\cdot}$, τ for a two-mass system with two collision pairs for which the collisionless motions are described by Eqs. [4]

$$y^{\cdot\cdot} = -\sin\tau, \quad |y| < d \qquad (4.1)$$

From Eq. (4.1) it follows that at the end points of the slippage trajectory segments (where $y = \mp d$, $y^{\cdot} = 0$, $y^{\cdot\cdot} = 0$) the value of $y^{\cdot\cdot\cdot} = -\cos\tau$ is extremal and equal to $+1$ on the collision interaction surface $y = -d$ and to $y^{\cdot\cdot\cdot} = -1$ on the surface $y = +d$. In this case conditions (2.1) and (2.2) become

$$\mp\cos\tau > 0, \quad \mp\cos\tau > \frac{9R-5}{12R}, \quad \mp\sin\tau < 0 \qquad (4.2)$$

$$0 < \mp y^{\cdot} < \frac{5(1-R)}{24R}\sin^2\tau$$

The minus sign refers to the surface $y = -d$ and the plus sign to the surface $y = +d$. For $R = 0.5$, for example, inequalities (4.2) yield the conditions

$$\mp\cos\tau > 0, \quad \mp\sin\tau < 0, \quad 0 < \mp y^{\cdot} < 0{,}21\sin^2\tau$$

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